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$\mathcal{M}$-theory on pp-waves with a holomorphic superpotential and its membrane and matrix descriptions

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Abstract: We study a new class of inhomogeneous pp-wave solutions with 8 unbroken supersymmetries in $D = 11$ supergravity. The 9 dimensional transverse space is Euclidean and split into 3 and 6 dimensional subspaces. The solutions have non-constant gauge flux, which are described in terms of an arbitrary holomorphic function of the complexified 6 dimensional space. The supermembrane and matrix theory descriptions are also provided and we identify the relevant supersymmetry transformation rules. The action also arises through a dimensional reduction of $\mathcal{N} = 1, D = 4$ supersymmetric Yang-Mills theory coupled to 3 gauge adjoint and chiral multiplets, whose interactions are determined by the holomorphic function of the supergravity solution now constituting the superpotential.

Keywords: Penrose limit and pp-wave background, M(atrix) Theories, M-Theory.

Contents

1. Introduction

Over the years it has become more evident that string theory, as a candidate of quantum gravity, and Yang-Mills theory are dual to each other [1]. One important line of progress has been made around the Matrix theory conjecture [2], which suggests that M-theory, the quantum completion of 11 dimensional supergravity theory in Minkowski background, is reduced to supersymmetric Yang-Mills quantum mechanics with SU(N) gauge group in the large N limit, when viewed in light-cone frame. There are two seemingly different ways to justify the Matrix theory conjecture. One is as a discretized supermembrane action [3], and the other is as the D0-brane dynamics which is believed to give a partonic description of M-theory when quantized along a light-like direction [2].

It is certainly desirable to extend the Matrix theory conjecture to more general backgrounds with less supersymmetry and smaller isometry groups, see e.g. [4, 5]. A natural way to explore is to turn on the gauge flux. Indeed, when one considers the maximally supersymmetric plane-wave solution [6], it is again possible to find the supermembrane action in light-cone gauge and the resulting Yang-Mills quantum mechanics is conjectured to give the corresponding Matrix theory description [7, 8]. This particular matrix model is usually called the BMN (Berenstein, Maldacena, and Nastase) matrix model, and thanks to the mass parameter set by the non-vanishing flux, one can perturbatively compute the energy spectrum [8, 9], unlike the original Matrix model in flat background. The existence of protected supermultiplets [9 − 11] turns out to be essential to verify the duality at the non-perturbative level, e.g. the dual modes of transverse M5-branes in the matrix model [12].

It is also noteworthy that the mass parameter of the Matrix theory can be traced back to the radius of S^3, when one puts the superconformal $\mathcal{N} = 4, D = 4$ Yang-Mills theory on $\mathbb{R} \times S^3$ for dimensional reduction [13].

It is straightforward to consider similar plane-wave solutions of $D = 11$ supergravity with constant flux and less supersymmetries [14 − 16]. One notable feature of such solutions is the so-called *supernumerary* supersymmetries, which mean that they preserve between 16 to 32 supersymmetries. It is possible for some of such backgrounds to identify the string/M-theory origin, for instance as intersecting M-brane configurations [17].

In this paper we attempt a further generalization and consider pp-waves with *non-constant* gauge fluxes. More specifically, we divide the 9 dimensional transverse space into a real 3 dimensional space $\mathbb{R}^3$ and a complex 3 dimensional subspace $\mathbb{C}^3$, and allow the configurations to depend only on the coordinates of $\mathbb{C}^3$ through a holomorphic function, which we call a superpotential. As a result the metric tensors will have $\mathrm{SO}(3) \times \mathrm{U}(1) \times \mathbb{R}$ isometry, where $\mathrm{U}(1)$ is the remaining invariance of the complex 3 dimensional space guaranteed by holomorphicity, and $\mathbb{R}$ denotes the null Killing vector. However, $\mathrm{U}(1)$ is generically broken for the entire solution when we also take the flux into account.

Inhomogeneous pp-waves with non-constant flux have been considered by several authors in similar settings. For 10 dimensional IIB supergravity, pp-waves on special holonomy manifolds are studied and it is shown that the Ramond-Ramond 5-form induces a superpotential on the light cone worldsheet Lagrangian [18], while the 3-forms are responsible for Killing vector potentials [19]. More recently, inspired by [20], supersymmetric Matrix models with so-called β-deformation are studied [21], and it is illustrated that the deformation superpotential of the discretized supermembrane action is given by inhomogeneous background fluxes which have a linear dependence on transverse coordinates. It is this discovery which motivated our research on pp-waves with a generic superpotential reported in this paper.

We establish the light-cone supermembrane action in our new inhomogeneous pp-wave configurations and write down the relevant matrix model action, which turns into the supermembrane action in the continuum large N limit. Because this matrix model has a generic superpotential, whose arguments are promoted to matrices, we encounter the usual matrix ordering ambiguity problem. We give an exposition on how this ambiguity is fixed by supersymmetry and the requirement to express the superpotential as a gauge singlet, i.e. a single or multi-trace operator.

The symmetry of our solutions and the existence of a holomorphic function suggests that these models should be naturally related to $\mathcal{N} = 1, D = 4$ super Yang-Mills theory with 3 chiral multiplets in the adjoint representation. It is verified explicitly through decomposition of the fermionic as well as the bosonic fields, and we identify the total superpotential of the Yang-Mills quantum mechanics.

This paper is organised as follows. In section 2, we present the pp-wave solutions of 11 dimensional supergravity we will be dealing with in this paper. It is also shown that these backgrounds in general allow 8 nontrivial solutions to the Killing spinor equation. In section 3, we consider the light-cone action of supermembranes in the pp-wave background, and show how it is reduced to a gauge theory of area-preserving diffeomorphisms. In section

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4 we provide the Yang-Mills quantum mechanics action which is obtained via the usual discretization method of replacing the Poisson brackets with commutators [22, 3]. In section 5, we establish that our supermembrane/matrix actions can be also expressed as $\mathcal{N}=1$ Yang-Mills theory with 3 interacting chiral multiplets, and identify the total superpotential. In section 6 we conclude with brief discussions.

2. A class of supersymmetric pp-wave solutions in $D = 11$ supergravity

Let us start by presenting the supergravity solutions we will study in this paper. Readers are referred to, for instance [15], for conventions of $D = 11$ supergravity.

Most generally, by a pp-wave in 11 dimensional supergravity we mean the following type of configurations

$$ds^2 = 2dx^- dx^+ + H(x^+, x^M)(dx^+)^2 + \sum_{M=1}^{9} (dx^M)^2, \tag{2.1}$$

$$F^{(4)} = dx^+ \wedge \phi(x^+, x^M). \tag{2.2}$$

The above ansatz is greatly simplifying and one can easily verify that the only nontrivial component of the Einstein equation gives

$$\nabla^2 H = -\frac{1}{6}\phi_{MNP}\phi^{MNP}, \tag{2.3}$$

where the Laplacian is taken in the 9 dimensional transverse space. One of course has to consider the flux equation of motion and the Bianchi identity for $F^{(4)}$, so we demand ϕ is harmonic:

$$d\phi = 0, \quad d(*_9\phi) = 0. \tag{2.4}$$

In this paper we are interested in a rather special subclass of the general pp-waves given above. We first divide the 9 dimensional space into 3 dimensional and 6 dimensional subspaces, and assume H, ϕ can depend only on the 6 dimensional coordinates. We will find it convenient to employ complex coordinates for the 6 dimensional space. Let us call them $z_a, (a = 1, 2, 3)$, and $\bar{z}_{\bar{a}}$ are the complex conjugates. Now the 9 dimensional part of the metric is written as

$$ds_9^2 = \sum_{i=1}^{3}(dx^i)^2 + 2\sum_{a=1}^{3} dz^a d\bar{z}^{\bar{a}}, \tag{2.5}$$

and accordingly $\nabla^2 = 2\sum_{a=1}^{3} \partial_a \bar{\partial}_{\bar{a}}$.

The upshot is that we can obtain a large class of solutions which are reminiscent of $\mathcal{N} = 1, D = 4$ supersymmetric field theory. Our construction is as follows. Firstly, we choose ϕ as a primitive $(2, 1)$ form plus its complex conjugate, in the space spanned by z_a. Componentwise, one writes

$$\phi_{\bar{a}bc} = \partial_{\bar{a}}\partial_{\bar{d}}\overline{W}\epsilon^{\bar{d}}{}_{bc}, \tag{2.6}$$

and in the same way for the complex conjugate, $\phi_{a\bar{b}\bar{c}}$. Note that W is a holomorphic function of z^a, and we take the convention $\epsilon_{123} = \epsilon_{\bar{1}\bar{2}\bar{3}} = 1$ for the totally antisymmetric tensor. Primitivity means that a symplectic trace of ϕ is zero, i.e. $\phi_{\bar{a}ab} = 0$, implying ϕ is imaginary self-dual in 6 dimensions.

It can be easily confirmed that ϕ, as given above, is in fact closed and co-closed. One easily integrates eq. (2.6) and the 2-form potential ψ, with $d\psi = \phi$, is given in terms of a $(2,0)$ form,

$$\psi_{ab} = \epsilon^{\bar{c}}{}_{ab}\partial_{\bar{c}}\overline{W}. \tag{2.7}$$

Now we only need to check the equation eq. (2.3) with an appropriate choice of H. It is easily seen that eq. (2.3) is indeed satisfied with

$$H = -|\partial W|^2. \tag{2.8}$$

Having established that the configuration indeed satisfies the equations of motion, let us now consider the Killing spinor equations. For 11 dimensional pp-waves with constant flux, the Killing spinor equations have been studied in detail in ref. [14−16]. We closely follow the convention and the analysis of [15], which is repeated here to some extent for self-sufficiency.

For the pp-wave metric, it is convenient to choose the following frame

$$\begin{aligned} e^+ &= dx^+, && (2.9)\\ e^- &= dx^- + \frac{H}{2}dx^+, && (2.10)\\ e^M &= dx^M, && (2.11) \end{aligned}$$

then the only nonvanishing components of the spin connection are

$$\omega^{-M} = \frac{1}{2}\partial_M H dx^+. \tag{2.12}$$

From the $D = 11$ supersymmetry transformation rule, the invariance of the gravitino requires $\nabla_\mu\epsilon = \Omega_\mu\epsilon$, with

$$\Omega_\mu = \frac{1}{288}(\gamma_\mu{}^{\nu\rho\sigma\tau} - 8\delta^\nu_\mu\gamma^{\rho\sigma\tau})F^{(4)}_{\nu\rho\sigma\tau}. \tag{2.13}$$

From the pp-wave ansatz Ω_μ is reduced to

$$\begin{aligned} \Omega_+ &= -\frac{1}{12}\Theta(\gamma_-\gamma_+ + 1), && (2.14)\\ \Omega_- &= 0, && (2.15)\\ \Omega_M &= \frac{1}{24}(3\Theta\gamma_M + \gamma_M\Theta)\gamma_-, && (2.16) \end{aligned}$$

where $\Theta = \frac{1}{6}\phi_{MNP}\gamma^{MNP}$.

At this stage, it is convenient to introduce two SO(9) spinors, $\epsilon_\pm$, to describe the 11 dimensional Majorana spinor ϵ. We use the basis where γ_μ are 32×32 matrices and given as

$$\gamma_+ = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}, \qquad \gamma_- = \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}, \qquad \gamma_M = \begin{pmatrix} \Gamma_M & 0\\ 0 & -\Gamma_M\end{pmatrix}, \tag{2.17}$$

where $M = 1, \ldots, 9$ and Γ_M are SO(9) gamma matrices. We can in fact take Γ_M to be all real and symmetric. Now, if we accordingly decompose ϵ as

$$\epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} , \tag{2.18}$$

we have the following equations for $\epsilon_\pm$.

$$\begin{aligned} \partial_+ \epsilon_+ &= -\frac{1}{12} \Theta \epsilon_+ \,, & \partial_+ \epsilon_- &= -\frac{\sqrt{2}}{4} \partial\!\!\!/ H \epsilon_+ + \frac{1}{12} \Theta \epsilon_- \,, \\ \partial_- \epsilon_+ &= 0 \,, & \partial_- \epsilon_- &= 0 \,, \\ \partial_M \epsilon_+ &= 0 \,, & \partial_M \epsilon_- &= \frac{\sqrt{2}}{24} (3\Theta\Gamma_M + \Gamma_M \Theta) \epsilon_+ \,. \end{aligned} \tag{2.19}$$

With a slight abuse of the notation, we now re-defined $\Theta = \frac{1}{6}\Phi_{MNP}\Gamma^{MNP}$. We see that, in general we can first solve the equations for ϵ_+, then plug it into the equations for ϵ_-. A simple type of solutions, which are sometimes called *kinematic* supersymmetries, are given as follows: We set $\epsilon_+ = 0$, and demand ϵ_- is also constant and annihilated by Θ. Since our 3-form field ϕ is $(2, 1)$ and primitive, Θ annihilates any SU(3) singlet spinor, which satisfies the following projection rules

$$\Gamma_{12}\epsilon = \Gamma_{34}\epsilon = \Gamma_{56}\epsilon. \tag{2.20}$$

Let us denote hereafter, a constant and Majorana spinor of SO(9) satisfying eq. (2.20) as $\epsilon^{(0)}$. It is now obvious that

$$\epsilon_+ = 0, \quad \epsilon_- = \epsilon^{(0)}, \tag{2.21}$$

provides 4 linearly independent solutions of the Killing equation.

Now let us verify that our background configuration in fact allows 4 more supersymmetries with $\epsilon_+ \neq 0$. This is sometimes called *dynamical* supersymmetries, and are responsible for the supersymmetry of the supermembrane action or the associated super Yang-Mills action which will be derived in the remainder of this paper. We already know that the equation for ϵ_+ can be solved by any constant spinor if it is an SU(3) singlet. So we first set $\epsilon_+ = \epsilon^{(0)}$. After a little computation, one can verify that if we set

$$\epsilon_- = -\frac{\sqrt{2}}{8} \partial_a W \epsilon^{abc} \Gamma_{bc} \epsilon^{(0)} + c.c. \tag{2.22}$$

then the equations for ϵ_- are identically satisfied. Note that our Killing spinor solutions, kinematical and dynamical altogether, have no dependence on x^+.

3. The light-cone supermembrane description

We now wish to derive the light-cone gauge fixed action for a supermembrane propagating in the above pp-wave background. The structure of the (super)-membrane action in curved backgrounds has been analyzed in a number of works, including [23, 24]. We here briefly repeat this construction in an economic first order formalism.

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3.1 First order formalism

In order to fix the light-cone gauge it is advantageous to bring the Polyakov formulation of the membrane into a first order formulation. The bosonic membrane propagating in a general background geometry $G_{\mu\nu}(X)$ and 3-form potential $C_{\mu\nu\rho}(X)$ with the membrane embedding coordinates $X^\mu = X^\mu(\tau, \sigma_1, \sigma_2)$ is given by

$$\begin{aligned} S = &-\frac{T}{2} \int d^3\xi \left(\gamma^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\, G_{\mu\nu}(X) - \sqrt{-h} \right) \\ &+ \frac{\kappa}{3!} \int d^3\xi\, \epsilon^{\alpha\beta\gamma}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\, \partial_\gamma X^\rho\, C_{\mu\nu\rho}(X), \end{aligned} \tag{3.1}$$

where $h_{\alpha\beta}$ is the world-volume metric and we have defined $\gamma^{\alpha\beta} = \sqrt{-h}\, h^{\alpha\beta}$ with $\det \gamma = -\sqrt{-h}$. T is the membrane tension and $\kappa = \pm 1$. This 'Polyakov' form of the action is equivalent to the first order formulation

$$\begin{aligned} S' = \int d^3\xi \Big[P_\mu\, \dot{X}^\mu + \frac{1}{2T\gamma^{00}} \Big\{ & P_\mu\, P_\nu\, G^{\mu\nu}(X) + T^2\, \gamma^{0r}\, \partial_r X^\mu\, \gamma^{0s}\, \partial_s X^\nu\, G_{\mu\nu} \\ & - \kappa\, P_\mu\, G^{\mu\nu}\, C_{\nu\rho\kappa}\, \epsilon^{\rm rs}\, \partial_r X^\rho\, \partial_s X^\kappa \Big\} - \frac{T}{2} \Big(\gamma^{\rm rs}\, \partial_r X^\mu\, \partial_s X^\nu\, G_{\mu\nu}(X) \\ & + \det \gamma \Big) + \frac{\gamma^{0r}}{\gamma^{00}}\, P_\mu\, \partial_r X^\mu \Big]. \end{aligned} \tag{3.2}$$

Here $r, s = 1, 2$ denote the space-like directions on the membrane world-volume. One checks that plugging back into S' the solution of the algebraic field equations for P_μ yields S. The equations of motion for the non-dynamical $\gamma^{\alpha\beta}$ give rise to the constraints of the theory. We now proceed by choosing the gauge condition $\gamma^{0r} = 0$ which turns its associated constraint equation into

$$P_\mu\, \partial_r X^\mu = 0\,. \tag{3.3}$$

This is the analogue of the level matching condition in string theory. Furthermore the equation of motion for $\gamma^{\rm rs}$ can be solved to give

$$\gamma^{\rm rs} = \frac{1}{\gamma^{00}} \begin{pmatrix} -\partial_2 X \cdot \partial_2 X & \partial_1 X \cdot \partial_2 X \\ \partial_1 X \cdot \partial_2 X & -\partial_1 X \cdot \partial_1 X \end{pmatrix}. \tag{3.4}$$

Inserting this result into (3.2) yields the first order form of the action

$$\begin{aligned} S' = \int d^3\xi \Big[P_\mu\, \dot{X}^\mu + \frac{1}{2T\gamma^{00}} \Big\{ & P_\mu\, G^{\mu\nu}(X) \Big(P_\nu - \kappa\, C_{\nu\rho\kappa}(X)\, \{X^\rho, X^\kappa\} \Big) \\ & + \frac{T^2}{2}\, \{X^\mu, X^\nu\}\, \{X^\rho, X^\kappa\}\, G_{\mu\rho}(X)\, G_{\nu\kappa}(X) \Big\} \Big], \end{aligned} \tag{3.5}$$

with the usual definition of the Poisson bracket $\{X^\mu, X^\nu\} := \epsilon^{\rm rs}\, \partial_r X^\mu\, \partial_s X^\nu$. This formulation of the theory is a suitable starting point for a light-cone gauge.

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3.2 Gauge fixed action

We now impose the light-cone gauge conditions

$$X^+ = \tau, \qquad P_- = 1\,. \tag{3.6}$$

Let us then specialize to the background of our inhomogenous pp-wave ansatz (2.1) ($r, s, t = 1, \ldots, 6$, $M, N = 1, \ldots, 9$)

$$\begin{aligned} G_{++} = &\quad H(X^r), \quad G_{+-} = 1, \quad G_{--} = 0, \quad G_{MN} = \delta_{MN} = G^{MN}, \quad C_{+rs} \neq 0, \\ G^{--} = &-H(X^r), \quad G^{+-} = 1, \quad G^{++} = 0. \end{aligned} \tag{3.7}$$

The light-cone Hamiltonian $-P_+$ now follows from solving the equations of motions for γ^{00} emerging from (3.5) for this specific background. Due to the gauge choices (3.6) many terms cancel and one finds

$$\mathcal{H}_{\rm LC} = -P_+ = \frac{1}{2}\left(P_M^2 - H(X^r) - \kappa\, C_{+rs}(X^t)\,\{X^r, X^s\} + \frac{T^2}{2}\,(\{X^M, X^N\})^2\right). \tag{3.8}$$

Alternatively one may consider the gauge-fixed second order form of the action which is obtained from (3.5) upon reinserting the solution of the equations of motion for the transverse momenta and the above P_+. One then finds

$$\mathcal{L}_{\rm GF} = \frac{1}{2}\,\dot{X}_M^2 + \frac{1}{2}\,H(X^r) + \frac{\kappa}{2}\,C_{+rs}(X^t)\,\{X^r, X^s\} - \frac{T^2}{4}\,(\{X^M, X^N\})^2\,. \tag{3.9}$$

This constitutes the light-cone bosonic membrane action in the background (3.7).

We now turn to the fermionic sector which has been neglected so far. The linear couplings to the background fields are known for the complete light-cone supermembrane from the supermembrane vertex operator construction of [25]. From this we infer that next to the usual flat-space fermion structure there is only one additonal Yukawa-type interaction term coupling to the three-form. We then have (setting $T = \kappa = 1$) the gauge fixed supermembrane lagrangian

$$\begin{aligned} \mathcal{L}_{\rm APD} =& \frac{1}{2}\,(D_\tau X_M)^2 + \frac{1}{2}\,H(X^r) + \frac{1}{2}\,C_{+rs}(X^t)\,\{X^r, X^s\} - \frac{1}{4}\,(\{X^M, X^N\})^2 \\ &+ i\,\theta^\dagger\,D_\tau\theta + i\,\theta^\dagger\Gamma_M\,\{X^M, \theta\,\} - \frac{i}{8}\,(\partial_r\,C_{+st})\,\theta^\dagger\Gamma^{\rm rst}\theta\,, \end{aligned} \tag{3.10}$$

where θ are SO(9) Majorana spinors. In (3.10) we have also promoted the τ derivatives to covariant ones $D_\tau\,\mathcal{O} = \partial_\tau\mathcal{O} - \{\omega, \mathcal{O}\}$, where ω is the gauge field of area preserving diffeomorphisms (APD) whose equations of motion give rise to the remaining 'level-matching' contraint equations (3.3). This formulation of the supermembrane allows for a SU(N) matrix regularisation in the usual fashion known from a flat space background [3] modulo ordering ambiguities.

Next we go to a complex basis in the bosonic SO(6) sector: $X^1, \ldots, X^6 \to Z^a, \bar{Z}^{\bar{a}}$ with $a = 1, 2, 3$ and $\bar{a} = \bar{1}, \bar{2}, \bar{3}$ explicitly we take

$$\begin{aligned} Z^1 &= \frac{1}{\sqrt{2}}\,(X^1 + i\,X^2), \qquad & \bar{Z}^{\bar{1}} &= \frac{1}{\sqrt{2}}\,(X^1 - i\,X^2) = (Z^1)^\dagger, \qquad & &\text{etc.} \\ X^1 &= \frac{1}{\sqrt{2}}\,(Z^1 + \bar{Z}^{\bar{1}}), \qquad & X^2 &= -\frac{i}{\sqrt{2}}\,(Z^1 - \bar{Z}^{\bar{1}}), \qquad & &\text{etc.} \end{aligned} \tag{3.11}$$

and the metric reads, now letting $M = (a, \bar{a}, i)$,

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_{a\bar{b}} & 0 \\ \delta_{\bar{b}a} & 0 & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix} . \tag{3.12}$$

Note that depending on the context we employ a real or complex notation for the SO(6) indices, *i.e.* $M = (a, \bar{a}, i)$ *or* $M = (r, i)$, however maintaining the same symbol X^M for the embedding coordinates.[1] The distinction should be clear, however, from the context.

The background field data in this language from the supergravity analysis of section 2 is

$$H(Z, \bar{Z}) = -\partial_a W(Z)\, \partial_{\bar{a}} \overline{W}(\bar{Z}) \,, \qquad C_{+ab} = \epsilon_{\rm dab}\, \partial_{\bar{d}} \overline{W}(\bar{Z}) \,, \qquad C_{+\bar{a}\bar{b}} = \epsilon_{\bar{d}\bar{a}\bar{b}}\, \partial_d W(Z) \,. \tag{3.13}$$

In (3.10) the SO(9) gamma matrices Γ^M are 16×16 nine-dimensional (with the indices $M = a, \bar{a}, i$) and satisfy the Clifford algebra,

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN} \,. \tag{3.14}$$

With the charge conjugate matrix C,

$$(\Gamma^M)^T = (\Gamma_M)^* = C\Gamma^M C^{-1} \,, \qquad C = C^T \,, \tag{3.15}$$

the 16-component spinor θ satisfies the Majorana condition

$$\theta^\dagger = \theta^T C \,. \tag{3.16}$$

We then have the following form of the gauge fixed supermembrane lagrangian

$$\begin{aligned} \mathcal{L}_{\rm APD} =& \frac{1}{2}\, D_\tau X^M\, D_\tau X_M - \frac{1}{4}\, \{X^M, X^N\}\{X_M, X_N\} + i\, \theta^\dagger\, D_\tau \theta + i\, \theta^\dagger \Gamma_M\, \{X^M, \theta\} \\ & - \frac{1}{2}\, \partial_a W\, \partial_{\bar{a}} \overline{W} + \frac{1}{2}\, \epsilon_{\rm dab}\, \partial_{\bar{d}} \overline{W}\, \{Z^a, Z^b\} + \frac{1}{2}\, \epsilon_{\bar{d}\bar{a}\bar{b}}\, \partial_d W\, \{\bar{Z}^{\bar{a}}, \bar{Z}^{\bar{b}}\} \\ & - \frac{i}{8}\, \varepsilon_{\bar{d}\bar{a}\bar{b}}\, \partial_c \partial_d W\, \theta^\dagger \Gamma_{\bar{c}ab} \theta - \frac{i}{8}\, \varepsilon_{dab}\, \partial_{\bar{c}} \partial_{\bar{d}} \overline{W}\, \theta^\dagger \Gamma_{c\bar{a}\bar{b}} \theta \,. \end{aligned} \tag{3.17}$$

In the sequel we shall show that this action also arises from dimensional reduction of $\mathcal{N} = 1$ super Yang-Mills coupled to three chiral matter multiplets with a superpotential dictated by $W(Z)$. Before doing so we will present the matrix theory version of this supermembrane theory and state the supersymmetry transformations.

4. The $\mathcal{M}$-theory matrix model description

The standard matrix discretization procedure of the supermembrane action (3.17) replaces the embedding coordinates by $N \times N$ matrices and the Poisson-brackets by commutators, i.e.

$$X^M(\tau, \sigma_1\, \sigma_2) \longrightarrow (X^M)_{ij}(\tau) \,, \qquad \{\cdot\,, \cdot\} \longrightarrow i\, [\,\cdot\,, \cdot\,] \,. \tag{4.1}$$

[1] The embedding coordinates satisfies then the following reality condition

$$(X^M)^\dagger = \eta_{MN} X^N = X_M \qquad \text{where} \quad X^M = (Z^a, \bar{Z}^{\bar{a}}, X^i) \,.$$

However, writing down the supersymmetric matrix model associated to (3.17) is nontrivial due to ordering ambiguities of the matrices. Here we present our result first, and then discuss the subtleties involved. We claim that the following $\mathcal{M}$-theory matrix model,

$$\begin{aligned}\mathcal{L}_{\rm MM} =&\mathrm{Tr}\Big(\frac{1}{2}D_tX^MD_tX_M+\frac{1}{4}[X^M,X^N][X_M,X_N]+i\theta^\dagger D_t\theta-\theta^\dagger\Gamma^M[X_M,\theta]\Big)\\&+\frac{1}{2}\,\mathrm{Tr}\Big(i\epsilon_{\bar a\bar b}{}^c[\bar Z^{\bar a},\bar Z^{\bar b}]\partial_c W+i\epsilon_{ab}{}^{\bar c}[Z^a,Z^b]\bar\partial_{\bar c}\overline{W}-\partial_aW\partial^a\overline{W}\Big)\\&-\frac{i}{8}\,\mathrm{Tr}\Big(\theta^\dagger\Gamma^a\Gamma^{\bar1\bar2\bar3}\Gamma^b\partial_a\mathrm{Tr}\left(\theta\partial_bW\right)+\theta^\dagger\Gamma^{\bar a}\Gamma^{123}\Gamma^{\bar b}\bar\partial_{\bar a}\mathrm{Tr}\left(\theta\bar\partial_{\bar b}\overline{W}\right)\Big)\,,\end{aligned}\tag{4.2}$$

enjoys four dynamical as well as four kinematical supersymmetries. Note that the Lagrangian above is determined by an arbitrary holomorphic superpotential W which is a scalar function of 3 hermitian matrices Z_a. It goes over to the membrane lagrangian (3.17) in the $N\to\infty$ limit upon replacing commutators by Poisson brackets. While this is rather obvious for the first two lines of (4.3), it is not much so for the Yukawa-type terms of the last line of (3.17). In order to make the comparison we note that, upon ignoring the ordering of the matrix valued θ and Z fields, we have

$$\begin{aligned}\mathrm{Tr}\Big(\theta^\dagger\Gamma^a\Gamma^{\bar1\bar2\bar3}\Gamma^b\partial_a\mathrm{Tr}\left(\theta\partial_bW\right)\Big)&\to\partial_e\partial_dW\,\theta^\dagger\Gamma^e\Gamma^{\bar1\bar2\bar3}\Gamma^d\theta=\partial_e\partial_dW\,\theta^\dagger\Gamma^e(\frac{1}{6}\epsilon_{\bar a\bar b\bar c}\Gamma^{\bar a\bar b\bar c})\Gamma^d\theta\\&=\epsilon_{\bar d\bar a\bar b}\partial_d\partial_cW\,\theta^\dagger\Gamma_{\bar cab}\theta\end{aligned}\tag{4.3}$$

matching the corresponding terms in (3.17).

W is an arbitrary U(N) singlet and holomorphic in Z_a, and $\overline{W}$ is the complex conjugate $W^\dagger$. More explicitly, W is a function of traces, like $\mathrm{Tr}(Z^{a_1}Z^{a_2}\cdots Z^{a_n})$, so we allow multi-traces, for instance. ∂_aW is matrix-valued as we suppress the matrix indices in our presentation. The time derivative is the gauge covariant one such as $D_tX^i=\frac{\mathrm{d}}{\mathrm{d}t}-i[A_0,X^i]$, A_0 is the matrix field corresponding to the APD gauge field ω in (3.17).

The $4+4$ supersymmetries are realized as

$$\begin{aligned}\delta A_0&=i\theta^\dagger\varepsilon\,,\qquad\delta X^M=i\theta^\dagger\Gamma^M\varepsilon\,,\\\delta\theta&=\frac{1}{2}\left(\Gamma^MD_tX_M-\frac{i}{2}\Gamma^{MN}[X_M,X_N]+\frac{1}{4}\Gamma^a\Gamma^{\bar1\bar2\bar3}\partial_aW+\frac{1}{4}\Gamma^{\bar a}\Gamma^{123}\bar\partial_{\bar a}\overline{W}\right)\varepsilon+\eta\,.\end{aligned}\tag{4.4}$$

The supersymmetry parameters ε and η are Majorana spinors $\varepsilon^\dagger=\varepsilon^TC$, $\eta^\dagger=\eta^TC$ and satisfy the following projection property

$$\varepsilon=P\varepsilon\,,\qquad\eta=P\,\eta\,,\qquad P=\tfrac{1}{8}(\Gamma^{123}\Gamma^{\bar3\bar2\bar1}+\Gamma^{\bar3\bar2\bar1}\Gamma^{123})\,.\tag{4.5}$$

Essentially the projector P leaves only the SU(3) singlet sector,[2] which has four nontrivial components, since $P=P^2=P^\dagger$ and tr$P=4$. We thus have four dynamical supersymmetries parametrzed by ε and four kinematical supersymmetries parametrizes η intact, matching the supergravity picture.

[2] P decomposes further into two orthogonal projections, $P=P_++P_-$ where $P_+=\frac{1}{8}\Gamma^{123}\Gamma^{\bar3\bar2\bar1}$, $P_-=\frac{1}{8}\Gamma^{\bar3\bar2\bar1}\Gamma^{123}$ satisfying $P_\pm=P_\pm^2=P_\pm^\dagger$, $P_+P_-=P_-P_+=0$ and tr$P_\pm=2$. We furthermore note the identities $\Gamma^{\bar1\bar2\bar3}\,\Gamma^a\,P=0$ and $\Gamma^{123}\,\Gamma^{\bar a}\,P=0$.

In order to verify the supersymmetry invariance it is noteworthy that the following terms, cubic in θ, vanish identically

$$\mathrm{Tr}\Big[\theta^\dagger\Gamma^a\varepsilon\partial_a\mathrm{Tr}\Big(\theta^\dagger\Gamma^b\Gamma^{\bar{1}\bar{2}\bar{3}}\Gamma^c\partial_b\mathrm{Tr}(\theta\partial_c W)\Big)\Big] = 0\,, \tag{4.6}$$

since it essentially corresponds to anti-symmetrizing three of two-component spinor indices. Other useful identities are[3]

$$[Z^a, \partial_a W] = 0\,, \qquad [\bar{Z}^{\bar{a}}, \bar{\partial}_{\bar{a}}\overline{W}] = 0\,. \tag{4.7}$$

Having established the supermembrane and matrix theory description of our $\mathcal{M}$-theory pp-wave background, we now proceed to investigate how these actions can be reexpressed as dimensional reduction of four-dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theory coupled to three chiral matter supermultiplets.

5. The $\mathcal{N}=1$ supersymmetry description

Our discussions so far make it evident that the supermembrane or matrix theory can be alternatively understood as the dimensional reduction of $\mathcal{N}=1, D=4$ super Yang-Mills theory to one dimension. In this section we start from the supermembrane action and rephrase it in a way where $\mathcal{N}=1$ symmetry is more manifest. We first show that in the bosonic sector the interactions are correctly given by the F-term and D-term potentials from the true superpotential which includes the familiar cubic superpotential of $\mathcal{N}=4$ theory, in addition to the superpotential of the supergravity background. We then proceed to decompose the SO(9) Majorana spinor into 4 copies of 2-component Weyl spinors, and in particular check that the Yukawa couplings are determined by the superpotential, just as one would expect from $\mathcal{N}=1$ supersymmetry.

5.1 The superpotential

The bosonic part of the gauge-fixed supermembrane or gauge theory of area-preserving diffeomorphism Lagrangian reads (3.10)

$$\begin{aligned}\mathcal{L}_{GF,bos} &= \frac{1}{2}\,(D_\tau X_M)^2 - V(X),\\ \text{with}\quad V(X) &= -\frac{1}{2}\,H(X^r) - \frac{1}{2}\,C_{+rs}\,\{X^r,X^s\} + \frac{1}{4}\,(\{X^M,X^N\})^2\,.\end{aligned} \tag{5.1}$$

[3]This can be checked by noting

$$[Z^a, \partial_a\mathrm{Tr}(Z^{b_1}Z^{b_2}\cdots Z^{b_n})] = \sum_{l=1}^{n}\Big(Z^{b_l}\cdots Z^{b_n}Z^{b_1}\cdots Z^{b_{l-1}} - Z^{b_{l+1}}\cdots Z^{b_n}Z^{b_1}\cdots Z^{b_l}\Big) = 0\,.$$

An analogue identity holds for the holomorphic superpotential in the supermembrane action in terms of Poisson bracket, $\{Z^a, \partial_a W(Z)\} = 0$.

We note the decomposition of the last piece of the scalar potential in the pure SO(6) sector ($r, s = 1, \ldots, 6$)

$$
\begin{aligned}
\frac{1}{4}\int d^2\sigma\,(\{X^r,X^s\})^2 &= \int d^2\sigma\left(\{Z^a,Z^b\}\{\bar{Z}^{\bar{a}},\bar{Z}^{\bar{b}}\}-\frac{1}{2}(\{Z^a,\bar{Z}^{\bar{a}}\})^2\right)\\
&= \int d^2\sigma\left(\frac{1}{2}\,\epsilon_{\rm dab}\,\{Z^a,Z^b\}\,\epsilon_{\bar{d}\bar{a}\bar{b}}\,\{\bar{Z}^{\bar{a}},\bar{Z}^{\bar{b}}\}-\frac{1}{2}(\{Z^a,\bar{Z}^{\bar{a}}\})^2\right), \qquad (5.2)
\end{aligned}
$$

where use of Jacobi's identity for the Poisson brackets has been made. With the help of this we can now rewrite the scalar potential $V(X^i, Z^a, Z^{\bar{a}})$ into F-term and D-term pieces along with the SO(3) and SO(6) mixed contributions ($m = 1, 2, 3$):

$$
\begin{aligned}
V(X,Z,\bar{Z}) =& \frac{1}{2}\left(\partial_d W-\epsilon_{\rm dab}\,\{Z^a,Z^b\}\right)\left(\partial_{\bar{d}}\overline{W}-\epsilon_{\bar{d}\bar{a}\bar{b}}\,\{\bar{Z}^{\bar{a}},\bar{Z}^{\bar{b}}\}\right)-\frac{1}{2}(\,\{Z^a,\bar{Z}^{\bar{a}}\}\,)^2\\
&+\frac{1}{4}(\{X^i,X^j\})^2+\{X^i,Z^a\}\,\{X^i,\bar{Z}^{\bar{a}}\}\,. \qquad (5.3)
\end{aligned}
$$

Hence the F-term piece of the potential is governed by the holomorphic superpotential

$$
\mathcal{W}(Z^1,Z^2,Z^3)=\int d^2\sigma\left(W(Z^1,Z^2,Z^3)-\frac{1}{3}\,\epsilon_{abc}\,Z^a\,\{Z^b,Z^c\}\right). \qquad (5.4)
$$

5.2 The fermionic terms

We now turn to a rewriting of the SO(9) spinors in (3.17) or respectively (4.3) in an SO(3) $\times$ SO(6) split following the conventions of [13], appendix A. For this we decompose the Dirac matrices according to ($i = 1, 2, 3, r = 1, \ldots, 6$)

$$
\Gamma_i=\begin{pmatrix}-\sigma_i\otimes\mathbf{1}_4 & 0\\ 0 & \sigma_i\otimes\mathbf{1}_4\end{pmatrix},\quad \Gamma_r=\begin{pmatrix}0 & \mathbf{1}_2\otimes\rho_r\\ \mathbf{1}_2\otimes\rho_r^\dagger & 0\end{pmatrix}, \qquad (5.5)
$$

where σ_i are the three Pauli matrices and the 4×4 matrices ρ_r and $\rho_r^\dagger$ satisfy

$$
\rho_r\,\rho_s^\dagger+\rho_s\,\rho_r^\dagger=\rho_r^\dagger\,\rho_s+\rho_s^\dagger\,\rho_r=2\,\delta_{\rm rs}\,\mathbf{1}_4\,. \qquad (5.6)
$$

The charge conjugation matrix in this representation is given by

$$
C_9=\begin{pmatrix}0 & \epsilon^{\alpha\beta}\otimes\mathbf{1}_4\\ -\epsilon_{\dot{\alpha}\dot{\beta}}\otimes\mathbf{1}_4 & 0\end{pmatrix}, \qquad (5.7)
$$

allowing one to write the spinor as[4]

$$
\theta=\begin{pmatrix}\theta_{\alpha A}\\ \bar{\theta}^{\dot{\alpha}}_A\end{pmatrix},\qquad \theta^\dagger=\theta^T\,C_9=(\bar{\theta}_{\dot{\alpha}A}\;\;\theta^\alpha_A)\,,\qquad \alpha=1,2\,,\;\;A=1,\ldots,4\,, \qquad (5.8)
$$

where $\theta_{\alpha A}$ and $\bar{\theta}_{\dot{\alpha}\,A}$ are now four 2-component Weyl spinors respectively. In terms of these the Yukawa couplings to the SO(6) scalars X^r may be reexpressed as

$$
\begin{aligned}
\{\theta^\dagger\Gamma_r,\theta\}\,X^r &= \{\theta_A,\theta_B\}\,(\rho_r^\dagger)_{AB}\,X^r+\{\bar{\theta}_A,\bar{\theta}_B\}\,(\rho_r)_{AB}\,X^r \qquad (5.9)\\
&= \{\theta_A,\theta_B\}\Big[(\Omega_a)_{AB}Z^a+(\Omega_{\bar{a}})_{AB}\bar{Z}^{\bar{a}}\Big]+\{\bar{\theta}_A,\bar{\theta}_B\}\Big[(\overline{\Omega}_a)_{AB}Z^a+(\overline{\Omega}_{\bar{a}})_{AB}\bar{Z}^{\bar{a}}\Big]\,,
\end{aligned}
$$

[4] We use the standard index free Weyl spinor notation with the convention: $\lambda\psi := -\varepsilon^{\alpha\beta}\lambda_\alpha\psi_\beta = \psi\lambda$ and $\bar{\lambda}\bar{\psi} := \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\alpha}}\bar{\psi}_{\dot{\beta}} = \bar{\psi}\bar{\lambda}$. Moreover $i(\sigma^2)^{\alpha\beta} = \epsilon^{\alpha\beta}$ and $(\lambda_\alpha)^* = \bar{\lambda}_{\dot{\alpha}}$.

where we have introduced the 4×4 matrices Ω_a and $\Omega_{\bar{a}}$ ($a = 1,2,3; \bar{a} = \bar{1},\bar{2},\bar{3}$) via

$$\begin{aligned}
\Omega_1 &= \frac{1}{\sqrt{2}}(\rho_1 - i\rho_2)\,, & \Omega_{\bar{1}} &= \frac{1}{\sqrt{2}}(\rho_1 + i\rho_2)\,, & &\text{etc.} \\
\overline{\Omega}_1 &= \frac{1}{\sqrt{2}}(\rho_1^\dagger - i\rho_2^\dagger)\,, & \overline{\Omega}_{\bar{1}} &= \frac{1}{\sqrt{2}}(\rho_1^\dagger + i\rho_2^\dagger)\,, & &\text{etc.}
\end{aligned} \tag{5.10}$$

which satisfy

$$\begin{aligned}
&\Omega_{\bar{a}}\overline{\Omega}_b + \Omega_b\overline{\Omega}_{\bar{a}} = \overline{\Omega}_{\bar{a}}\Omega_b + \overline{\Omega}_b\Omega_{\bar{a}} = 2\,\eta_{\bar{a}b}\,\mathbf{1}_4\,, \\
&\Omega_a\overline{\Omega}_b + \Omega_b\overline{\Omega}_a = \overline{\Omega}_a\Omega_b + \overline{\Omega}_b\Omega_a = 0\,, \\
&\Omega_{\bar{a}}\overline{\Omega}_{\bar{b}} + \Omega_{\bar{b}}\overline{\Omega}_{\bar{a}} = \overline{\Omega}_{\bar{a}}\Omega_{\bar{b}} + \overline{\Omega}_{\bar{b}}\Omega_{\bar{a}} = 0\,.
\end{aligned} \tag{5.11}$$

It is useful to employ a definite representation for the antisymmetric ρ_r matrices:

$$\begin{aligned}
\rho_1 &= -\mathbf{1} \otimes i\sigma_2\,, & \rho_2 &= -\sigma_3 \otimes \sigma_2\,, \\
\rho_3 &= -i\sigma_2 \otimes \sigma_3\,, & \rho_4 &= -\sigma_2 \otimes \mathbf{1}\,, \\
\rho_5 &= -i\sigma_2 \otimes \sigma_1\,, & \rho_6 &= -\sigma_1 \otimes \sigma_2\,.
\end{aligned} \tag{5.12}$$

In terms of these one finds

$$\Omega_a\,Z^a = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -Z^3 & Z^2 \\ 0 & Z^3 & 0 & -Z^1 \\ 0 & -Z^2 & Z^1 & 0 \end{pmatrix}, \qquad \Omega_{\bar{a}}\,\bar{Z}^{\bar{a}} = \sqrt{2}\begin{pmatrix} 0 & -\bar{Z}^{\bar{1}} & -\bar{Z}^{\bar{2}} & -\bar{Z}^{\bar{3}} \\ \bar{Z}^{\bar{1}} & 0 & 0 & 0 \\ \bar{Z}^{\bar{2}} & 0 & 0 & 0 \\ \bar{Z}^{\bar{3}} & 0 & 0 & 0 \end{pmatrix}. \tag{5.13}$$

We are led to identify θ_0 with the $\mathcal{N} = 1$ gluino λ and the remaining components with the SU(3) matter fermions ψ^a and $\bar{\psi}^{\bar{a}}$ in the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations via

$$\theta_A = (\lambda, \psi^1, \psi^2, \psi^3) \qquad \bar{\theta}_A = (\bar{\lambda}, \bar{\psi}^{\bar{1}}, \bar{\psi}^{\bar{2}}, \bar{\psi}^{\bar{3}})\,. \tag{5.14}$$

This leads us to the compact expressions

$$\begin{aligned}
\{\theta_A, \theta_B\}(\Omega_a)_{AB}\,Z^a &= -\sqrt{2}\,\varepsilon_{abc}\,Z^a\,\{\psi^b, \psi^c\}\,, \\
\{\theta_A, \theta_B\}(\Omega_{\bar{a}})_{AB}\,\bar{Z}^{\bar{a}} &= -2\sqrt{2}\,\bar{Z}^{\bar{a}}\,\{\lambda, \psi^a\} \\
\{\bar{\theta}_A, \bar{\theta}_B\}(\overline{\Omega}_a)_{AB}\,Z^a &= 2\sqrt{2}\,Z^a\,\{\bar{\lambda}, \bar{\psi}^{\bar{a}}\}\,, \\
\{\bar{\theta}_A, \bar{\theta}_B\}(\overline{\Omega}_{\bar{a}})_{AB}\,\bar{Z}^{\bar{a}} &= \sqrt{2}\,\varepsilon_{\bar{a}\bar{b}\bar{c}}\,\bar{Z}^{\bar{a}}\,\{\bar{\psi}^{\bar{b}}, \bar{\psi}^{\bar{c}}\}\,,
\end{aligned} \tag{5.15}$$

again suppressing the 2-component Weyl spinor indices. Upon inserting this into (5.9) reproduces the Yukawa couplings and part of the superpotential couplings of $\mathcal{N} = 1$ super Yang-Mills coupled to 3 chiral multiplets in the SU(3).

The remaining fermion term coupling to the three-form C_{+rs} of (3.10) (compare (3.17)) reads

$$\frac{i}{8}\,\partial_r C_{+st}\,\theta^\dagger\Gamma^{\mathrm{rst}}\theta = \frac{i}{8}\,\varepsilon_{\bar{d}\bar{a}\bar{b}}\,\partial_c\partial_d W\,\theta^\dagger\Gamma_{\bar{c}ab}\theta + \frac{i}{8}\,\varepsilon_{dab}\,\partial_{\bar{c}}\partial_{\bar{d}}\overline{W}\,\theta^\dagger\Gamma_{c\bar{a}\bar{b}}\theta\,. \tag{5.16}$$

In the $\mathrm{SO}(3) \times \mathrm{SO}(6)$ split the three-index Dirac matrices $\Gamma_{\bar{c}ab}$ take the form

$$\Gamma_{\bar{c}ab} = \begin{pmatrix} 0 & \mathbf{1}_2 \otimes \Omega_{\bar{c}ab} \\ \mathbf{1}_2 \otimes \overline{\Omega}_{\bar{c}ab} & 0 \end{pmatrix}, \quad \theta^\dagger\Gamma_{\bar{c}ab}\theta = \theta_A\theta_B\,(\Omega_{\bar{c}ab})_{AB} + \bar{\theta}_A\bar{\theta}_B\,(\overline{\Omega}_{\bar{c}ab})_{AB}\,, \tag{5.17}$$

with $\Omega_{\bar{c}ab} := \Omega_{[\bar{c}}\,\overline{\Omega}_a\,\Omega_{b]}$ and $\overline{\Omega}_{\bar{c}ab} := \overline{\Omega}_{[\bar{c}}\,\Omega_a\,\overline{\Omega}_{b]}$ antisymmetrized with unit weight. One then shows using the above representation that

$$\begin{aligned}\theta_A\theta_B\,(\Omega_{\bar{c}ab})_{AB} &= 2\sqrt{2}\,\psi^c\psi^d\,\varepsilon_{abd}\\ \bar{\theta}_A\bar{\theta}_B\,(\overline{\Omega}_{\bar{c}ab})_{AB} &= -2\sqrt{2}\,(\,\eta_{\bar{c}a}\,\bar{\lambda}\bar{\psi}^{\bar{b}} - \eta_{\bar{c}b}\,\bar{\lambda}\bar{\psi}^{\bar{a}}\,)\end{aligned} \tag{5.18}$$

Hence we have in (5.16)

$$\frac{i}{8}\,\varepsilon_{\bar{d}\bar{a}\bar{b}}\,\partial_c\partial_d W\,\theta_A\theta_B\,(\Omega_{\bar{c}ab})_{AB} = \frac{i}{\sqrt{2}}\,\partial_c\partial_d W\,\psi^c\psi^d\,, \tag{5.19}$$

the expected fermionic coupling in the matter sector to the holomorphic superpotential $W(Z^a)$, whereas the nonholomorphic second term in (5.16) drops out as it should:

$$\frac{i}{8}\,\varepsilon_{\bar{d}\bar{a}\bar{b}}\,\partial_c\partial_d W\,\bar{\theta}_A\bar{\theta}_B\,(\overline{\Omega}_{\bar{c}ab})_{AB} = \frac{i}{\sqrt{2}}\,\varepsilon_{\bar{d}\bar{c}\bar{b}}\,\partial_c\partial_d W\,\bar{\lambda}\bar{\psi}^{\bar{b}} = 0\,. \tag{5.20}$$

and the analogous terms for the hermitian conjugate contributions.

Upon collecting everything we indeed find an $\mathcal{N}=1$ super Yang-Mills model of area preserving diffeomorphisms coupled to three chiral multiplets transforming in the fundamental representation of SU(3) dimensionally reduced to one-time dimension:

$$\begin{aligned}\mathcal{L}_{GF,\ susy} =& \frac{1}{2}(D_\tau X^i)^2 - \frac{1}{4}\,(\{X^i,X^j\})^2 + D_\tau Z^a\,D_\tau\bar{Z}^{\bar{a}} - \{X^i,Z^a\}\,\{X^i,\bar{Z}^{\bar{a}}\}\\ &-\frac{1}{2}\partial_a\mathcal{W}(Z)\bar{\partial}_{\bar{a}}\overline{\mathcal{W}}(\bar{Z}) + \frac{1}{2}\,\{Z^a,\bar{Z}^{\bar{a}}\}^2\\ &+2i\lambda\,D_\tau\bar{\lambda} + 2i\,\lambda\sigma_i\,\{X^i,\bar{\lambda}\} + 2i\psi^a\,D_\tau\bar{\psi}^{\bar{a}} + 2i\,\psi^a\sigma_i\,\{X^i,\bar{\psi}^{\bar{a}}\}\\ &- i\,2\sqrt{2}\,\bar{Z}^{\bar{a}}\,\{\lambda,\psi^a\} + i\,2\sqrt{2}\,Z^a\,\{\bar{\lambda},\bar{\psi}^{\bar{a}}\}\\ &-\frac{i}{\sqrt{2}}\psi^a\partial_a\int d^2\sigma'\,\psi^b(\sigma')\frac{\partial}{\partial Z^b(\sigma')}\mathcal{W}(Z) + \frac{i}{\sqrt{2}}\bar{\psi}^{\bar{a}}\bar{\partial}_{\bar{a}}\int d^2\sigma'\,\bar{\psi}^{\bar{b}}(\sigma')\frac{\bar{\partial}}{\bar{\partial}\bar{Z}^{\bar{b}}(\sigma')}\overline{\mathcal{W}}(\bar{Z})\,.\end{aligned} \tag{5.21}$$

Here the holomorphic superpotential is given by an integral over the two-dimensional space like components of the membrane worldsheet

$$\mathcal{W}(Z^a) = \int d^2\sigma\,\left(\,W(Z^a) - \frac{1}{3}\,\epsilon_{abc}\,Z^a\,\{Z^b,Z^c\}\right)\,, \tag{5.22}$$

and any derivative acting on it, as $\partial_a\mathcal{W}$, must be understood as a functional derivative with respect to $Z^a(\sigma)$.

Now we are ready to obtain the Matrix theory action utilizing again the familiar discretisation procedure of replacing the Poisson brackets with matrix commutators:

$$\begin{aligned}\mathcal{L} =& \mathrm{Tr}\Bigg[\frac{1}{2}D_tX^iD_tX^i + \frac{1}{4}[X^i,X^j]^2 + D_tZ^aD_t\bar{Z}^{\bar{a}} + [X^i,Z^a][X^i,\bar{Z}^{\bar{a}}]\\ &-\frac{1}{2}\partial_a\mathcal{W}(Z)\bar{\partial}_{\bar{a}}\overline{\mathcal{W}}(\bar{Z}) - \frac{1}{2}[Z^a,\bar{Z}^{\bar{a}}]^2 + 2i\lambda D_t\bar{\lambda} - 2\lambda\sigma^i[X_i,\bar{\lambda}]\\ &+2i\psi^aD_t\bar{\psi}^{\bar{a}} - 2\psi^a\sigma^i[X_i,\bar{\psi}^{\bar{a}}] + 2\sqrt{2}\bar{Z}^{\bar{a}}[\lambda,\psi^a] - 2\sqrt{2}Z^a[\bar{\lambda},\bar{\psi}^{\bar{a}}]\\ &-\frac{i}{\sqrt{2}}\psi^a\partial_a\mathrm{Tr}\left(\psi^b\partial_b\mathcal{W}(Z)\right) + \frac{i}{\sqrt{2}}\bar{\psi}^{\bar{a}}\bar{\partial}_{\bar{a}}\mathrm{Tr}\left(\bar{\psi}^{\bar{b}}\bar{\partial}_{\bar{b}}\overline{\mathcal{W}}(\bar{Z})\right)\Bigg]\,.\end{aligned} \tag{5.23}$$

This constitutes our main result: $\mathcal{M}$-theory in a generalized pp-wave background with gauge flux described by a holomorphic function $W(Z^a)$ has its Matrix theory description as the dimensional reduction of $\mathcal{N}=1$, $D=4$, $\mathrm{U}(N)$ Yang-Mills theory coupled to three chiral supermultiplets with superpotential $\mathcal{W}(Z)$ given by

$$\mathcal{W}(Z) = W(Z) - \frac{i}{3}\epsilon_{abc}\mathrm{Tr}\Big(Z^a[Z^b, Z^c]\Big) . \tag{5.24}$$

In closing we would like to remark that the membrane or matrix theory actions found in the above are $\mathcal{N}=1$ supersymmetric irrespective of the form of $W(Z)$. However, different orderings of the fields appearing in $W(Z)$ lead to distinct marix models, but are equivalent in supergravity or supermembrane theory. In addition functions $W(Z)$ are possible containing Poisson-brackets or commutators of the holomorphic fields Z^a, which would have to be understood as being of non-geometric origin.

6. Discussions

In this paper we have considered a class of supersymmetric pp-wave solutions in 11 dimensional supergravity. The holomorphic function which describes the configuration is related to the superpotential of the Yang-Mills quantum mechanics which comes from the discretized supermembrane action in the relevant background.

Following the spirit of [2], it is natural to conjecture that the Yang-Mills quantum mechanics we have derived in this paper should provide a Matrix theory description of M-theory in the inhomogeneous pp-wave backgrounds. As an alternative to the supermembrane action, Matrix theory can be also obtained as discrete lightcone quantization (DLCQ) of M-theory [26]. In practice, one compactifies M-theory on a small circle and at the same time performs an infinite boost. The quantized lightlike momentum N is translated into the number of D0-branes through T-duality, and the large N Yang-Mills quantum mechanics of D0-branes gives the Matrix theory.

The generalization of DLCQ prescription of M-theory to nontrivial curved background was considered for instance in [27, 28], where the authors studied low energy dynamics of D0-branes in weakly curved backgrounds. As a simple but nontrivial example, one can consider the maximally supersymmetric 11 dimensional plane-wave and perform DLCQ of M-theory [8]. Although the scalar curvature of 11 dimensional background vanishes, the IIA configuration becomes singular when $H = G_{++} \to -4$. In [8] it is verified that in the small H approximation the D0-brane dynamics in the weakly curved background limit coincides with the regularized supermembrane action, or the BMN matrix theory [7].

One can also apply the method of [28] to those solutions we studied in this paper. First of all, one can easily verify that $H = G_{++}$ again translates into the scalar potential of the Yang-Mills theory, with Tseytlin's symmetrized trace prescription [29] for matrix fields. This way we can resolve the ambiguity of matrix ordering problem utilizing the microscopic description through open string excitations. The rest of the action should agree with the supermembrane prescription, since the various terms are related by $\mathcal{N}=1$ supersymmetry. Summarizing, although DLCQ description for generic pp-waves has a

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drawback of limited validity due to the singularity of IIA background, it in principle can fix the ordering problem of Matrix regularization we encounter in supermembrane action. It will be certainly interesting to further explore D0-brane dynamics in the T-dual background in IIA supergravity.

For the solutions we studied in this paper, the flux is turned on along the six dimensional subspace only, and the isometry group contains SO(3). It is thus natural to view the matrix model as originating from a four dimensional field theory. One might ask whether it is possible to turn on a constant flux on $\mathbb{R}^3$, without breaking SO(3). This is exactly the mass deformation which transforms the ordinary Matrix theory [2] into the BMN matrix model [7], and as a result there will be a cubic interaction term $\mathrm{Tr}X^1[X^2, X^3]$ in the matrix model.

It is elucidated in [13] that, as a four dimensional super Yang-Mills theory, the mass parameter is related to the choice of putting the field theory on $\mathbb{R} \times S^3$ instead of $\mathbb{R}^{1,3}$. Since this freedom relies on classical superconformal invariance, we expect it is not possible in general to have a constant flux in $\mathbb{R}^3$, unless W is cubic in Z. Among these the most interesting is probably the so-called β-deformation which is known to be exactly marginal as a 4 dimensional quantum field theory [20]. By β-deformation, the matrix commutator is replaced by

$$[X, Y] = XY - YX \longrightarrow e^{i\beta}XY - e^{-i\beta}YX, \tag{6.1}$$

for a constant β. In the context of Matrix theory, this deformation is considered in [21] and the stable membrane solutions of different topology are studied in the continuum limit.

It will be certainly very interesting to consider BPS objects with different dimensions in the matrix models described in this paper. For BMN matrix model, readers are referred to e.g. [30 – 32] for the study of BPS configurations.

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